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Research Article

Transformations of Difference Equations II

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This is an extension of the work done by Currie and Love (2010) where we studied the effect of applying two Crum-type transformations to a weighted second-order difference equation with non-eigenparameter-dependent boundary conditions at the end points. In particular, we now consider boundary conditions which depend affinely on the eigenparameter together with various combinations of Dirichlet and non-Dirichlet boundary conditions. The spectra of the resulting transformed boundary value problems are then compared to the spectra of the original boundary value problems.

1. Introduction

This paper continues the work done in [1], where we considered a weighted second-order difference equation of the following form:

$$c(n)y(n+1) - b(n)y(n) + c(n-1)y(n-1) = -c(n)\lambda y(n), \tag{1.1}$$

with $c(n) > 0$ representing a weight function and $b(n)$ a potential function.

This paper is structured as follows.

The relevant results from [1], which will be used throughout the remainder of this paper, are briefly recapped in Section 2.

In Section 3, we show how non-Dirichlet boundary conditions transform to affine λ-dependent boundary conditions. In addition, we provide conditions which ensure that the linear function (in λ) in the affine λ-dependent boundary conditions is a Nevanlinna or Herglotz function.

Section 4 gives a comparison of the spectra of all possible combinations of Dirichlet and non-Dirichlet boundary value problems with their transformed counterparts. It is shown that transforming the boundary value problem given by (2.2) with any one of the four combinations of Dirichlet and non-Dirichlet boundary conditions at the end points using (3.1) results in a boundary value problem with one extra eigenvalue in each case. This is done by considering the degree of the characteristic polynomial for each boundary value problem.

It is shown, in Section 5, that we can transform affine λ-dependent boundary conditions back to non-Dirichlet type boundary conditions. In particular, we can transform back to the original boundary value problem.

To conclude, we outline briefly how the process given in the sections above can be reversed.

2. Preliminaries

Consider the second-order difference equation (1.1) for $n = 0, \dots, m-1$ with boundary conditions

$$hy(-1) + y(0) = 0, \qquad Hy(m-1) + y(m) = 0, \tag{2.1}$$

where h and H are constants, see [2]. Without loss of generality, by a shift of the spectrum, we may assume that the least eigenvalue, λ_0, of (1.1), (2.1) is $\lambda_0 = 0$.

We recall the following important results from [1]. The mapping $y \mapsto \tilde{y}$ defined for $n = -1, \dots, m-1$ by $\tilde{y}(n) = y(n+1) - y(n)(u_0(n+1)/u_0(n))$, where $u_0(n)$ is the eigenfunction of (1.1), (2.1) corresponding to the eigenvalue $\lambda_0 = 0$, produces the following transformed equation:

$$\tilde{c}(n)\tilde{y}(n+1) - \tilde{b}(n)\tilde{y}(n) + \tilde{c}(n-1)\tilde{y}(n-1) = -\tilde{c}(n)\lambda\tilde{y}(n), \qquad n = 0, \dots, m-2, \tag{2.2}$$

where

$$\tilde{c}(n) = \frac{u_0(n)c(n)}{u_0(n+1)} > 0, \quad n = -1, \dots, m-1,$$
$$\tilde{b}(n) = \left[\frac{u_0(n)c(n)}{u_0(n+1)c(n+1)} - \frac{c(n-1)u_0(n-1)}{c(n)u_0(n)} + \frac{b(n)}{c(n)} - \lambda_0\right]\frac{u_0(n)c(n)}{u_0(n+1)}, \quad n = 0, \dots, m-2. \tag{2.3}$$

Moreover, y obeying the boundary conditions (2.1) transforms to $\tilde{y}$ obeying the Dirichlet boundary conditions as follows:

$$\tilde{y}(-1) = 0, \qquad \tilde{y}(m-1) = 0. \tag{2.4}$$

Applying the mapping $\tilde{y} \mapsto \hat{y}$ given by $\hat{y}(n) = \tilde{y}(n) - \tilde{y}(n-1)(z(n)/z(n-1))$ for $n = 0, \dots, m-1$, where $z(n)$ is a solution of (2.2) with $\lambda = \hat{\lambda}_0$, where $\hat{\lambda}_0$ is less than the least eigenvalue of (2.2), (2.4), such that $z(n) > 0$ for all $n = -1, \dots, m-1$, results in the following transformed equation:

$$\hat{c}(n)\hat{y}(n+1) - \hat{b}(n)\hat{y}(n) + \hat{c}(n-1)\hat{y}(n-1) = -\hat{c}(n)\lambda\hat{y}(n), \qquad n = 1, \dots m-2, \tag{2.5}$$

where, for $n = 0, \dots, m-1$,

$$\hat{c}(n) = \frac{z(n-1)\tilde{c}(n-1)}{z(n)},$$
$$\hat{b}(n) = \left[\frac{z(n-1)\tilde{c}(n-1)}{z(n)\tilde{c}(n)} + \frac{z(n)}{z(n-1)}\right]\frac{z(n-1)\tilde{c}(n-1)}{z(n)}. \tag{2.6}$$

Here, we take $\hat{c}(-1) = c(-1)$, thus $\hat{c}(n)$ is defined for $n = -1, \dots, m-1$.

In addition, $\tilde{y}$ obeying the Dirichlet boundary conditions (2.4) transforms to $\hat{y}$ obeying the non-Dirichlet boundary conditions as follows:

$$\hat{h}\hat{y}(-1) + \hat{y}(0) = 0, \qquad \widehat{H}\hat{y}(m-1) + \hat{y}(m) = 0, \tag{2.7}$$

where

$$\hat{h} = \left[\frac{\hat{c}(0)}{\hat{c}(-1)}\left(\frac{\tilde{b}(0)}{\tilde{c}(0)} - \frac{z(1)}{z(0)} - \frac{\hat{b}(0)}{\hat{c}(0)}\right)\right]^{-1},$$
$$\widehat{H} = \frac{\tilde{b}(m-2)}{\tilde{c}(m-2)} - \frac{\hat{b}(m-1)}{\hat{c}(m-1)} - \frac{z(m-2)\hat{c}(m-2)}{z(m-1)\hat{c}(m-1)}. \tag{2.8}$$

3. Non-Dirichlet to Affine

In this section, we show how $\tilde{v}$ obeying the non-Dirichlet boundary conditions (3.2), (3.13) transforms under the following mapping:

$$\hat{v}(n) = \tilde{v}(n) - \tilde{v}(n-1)\frac{z(n)}{z(n-1)}, \quad n = 0, \dots, m-1, \tag{3.1}$$

to give $\hat{v}$ obeying boundary conditions which depend affinely on the eigenparamter λ. We provide constraints which ensure that the form of these affine λ-dependent boundary conditions is a Nevanlinna/Herglotz function.

Theorem 3.1. *Under the transformation* (3.1), $\tilde{v}$ *obeying the boundary conditions*

$$\tilde{v}(-1) - \gamma\tilde{v}(0) = 0, \tag{3.2}$$

for $\gamma \neq 0$, *transforms to* $\hat{v}$ *obeying the boundary conditions*

$$\hat{v}(-1) = \hat{v}(0)(a\lambda + b), \tag{3.3}$$

where $a = \gamma k/[\hat{c}(-1)/\hat{c}(0) - k\gamma(\hat{c}(-1)/\hat{c}(0))]$, $b = [\hat{b}(0)/\hat{c}(0) - \gamma k(\hat{b}(0)/\hat{c}(0)) - \tilde{b}(0)/\tilde{c}(0) + \gamma(\tilde{c}(-1)/\tilde{c}(0)) + z(1)/z(0)]/[\hat{c}(-1)/\hat{c}(0) - \gamma k(\hat{c}(-1)/\hat{c}(0))]$, *and* $k = z(0)/z(-1)$. *Here,* $\hat{c}(-1) := \tilde{c}(-1)$ *and* $z(n)$ *is a solution of* (2.2) *for* $\lambda = \lambda_0$, *where* λ_0 *is less than the least eigenvalue of* (2.2), (3.2), *and* (3.13) *such that* $z(n) > 0$ *for* $n \in [-1, m-1]$.

Proof. The values of n for which $\widehat{v}$ exists are $n = 0, \ldots, m-1$. So to impose a boundary condition at $n = -1$, we need to extend the domain of $\widehat{v}$ to include $n = -1$. We do this by forcing the boundary condition (3.3) and must now show that the equation is satisfied on the extended domain.

Evaluating (2.5) at $n = 0$ for $\widehat{y} = \widehat{v}$ and using (3.3) gives the following:

$$\widehat{c}(0)\widehat{v}(1) - \widehat{b}(0)\widehat{v}(0) + \widehat{c}(-1)\widehat{v}(0)(a\lambda + b) = -\widehat{c}(0)\lambda\widehat{v}(0). \tag{3.4}$$

Also from (3.1) for $n = 1$ and $n = 0$, we obtain the following:

$$\begin{aligned} \widehat{v}(1) &= \widetilde{v}(1) - \widetilde{v}(0)\frac{z(1)}{z(0)}, \\ \widehat{v}(0) &= \widetilde{v}(0) - \widetilde{v}(-1)\frac{z(0)}{z(-1)}. \end{aligned} \tag{3.5}$$

Substituting (3.2) into the above equation yields

$$\widehat{v}(0) = \widetilde{v}(0)\left[1 - \gamma\frac{z(0)}{z(-1)}\right]. \tag{3.6}$$

Thus, (3.4) becomes

$$\widehat{c}(0)\left[\widetilde{v}(1) - \widetilde{v}(0)\frac{z(1)}{z(0)}\right] + \widetilde{v}(0)\left[1 - \gamma\frac{z(0)}{z(-1)}\right]\left[-\widehat{b}(0) + \widehat{c}(-1)(a\lambda + b) + \widehat{c}(0)\lambda\right] = 0. \tag{3.7}$$

This may be slightly rewritten as follows

$$\widetilde{v}(1) - \widetilde{v}(0)\left\{\frac{z(1)}{z(0)} - \left(1 - \gamma\frac{z(0)}{z(-1)}\right)\left[-\frac{\widehat{b}(0)}{\widehat{c}(0)} + \frac{\widehat{c}(-1)}{\widehat{c}(0)}b + \lambda\left(1 + \frac{\widehat{c}(-1)}{\widehat{c}(0)}a\right)\right]\right\} = 0. \tag{3.8}$$

Also from (2.2), with $n = 0$, together with (3.2), we have the following:

$$\widetilde{v}(1) - \widetilde{v}(0)\left[\frac{\widetilde{b}(0)}{\widetilde{c}(0)} - \frac{\widetilde{c}(-1)}{\widetilde{c}(0)}\gamma - \lambda\right] = 0. \tag{3.9}$$

Subtracting (3.9) from (3.8) and using the fact that $\widetilde{v}(0) \neq 0$ results in

$$\frac{\widetilde{b}(0)}{\widetilde{c}(0)} - \frac{\widetilde{c}(-1)}{\widetilde{c}(0)}\gamma - \lambda - \frac{z(1)}{z(0)} + \left(1 - \gamma\frac{z(0)}{z(-1)}\right)\left[-\frac{\widehat{b}(0)}{\widehat{c}(0)} + \frac{\widehat{c}(-1)}{\widehat{c}(0)}b\right] + \lambda\left(1 - \gamma\frac{z(0)}{z(-1)}\right)\left(1 + \frac{\widehat{c}(-1)}{\widehat{c}(0)}a\right) = 0. \tag{3.10}$$

Equating coefficients of λ on both sides gives the following:

$$a = \frac{\gamma k}{\widehat{c}(-1)/\widehat{c}(0) - k\gamma(\widehat{c}(-1)/\widehat{c}(0))} \tag{3.11}$$

and equating coefficients of λ^0 on both sides gives the following:

$$b = \frac{\widehat{b}(0)/\widehat{c}(0) - \gamma k\left(\widehat{b}(0)/\widehat{c}(0)\right) - \widetilde{b}(0)/\widetilde{c}(0) + \gamma(\widetilde{c}(-1)/\widetilde{c}(0)) + z(1)/z(0)}{\widehat{c}(-1)/\widehat{c}(0) - \gamma k(\widehat{c}(-1)/\widehat{c}(0))}, \tag{3.12}$$

where $k = z(0)/z(-1)$, and recall $\widehat{c}(-1) = \widetilde{c}(-1)$. □

Note that for $\gamma = 0$, this corresponds to the results in [1] for $b = -1/\widehat{h}$.

Theorem 3.2. *Under the transformation* (3.1), $\widetilde{v}$ *satisfying the boundary conditions*

$$\widetilde{v}(m-2) - \delta\widetilde{v}(m-1) = 0, \tag{3.13}$$

for $\delta \neq 0$, *transforms to* $\widehat{v}$ *obeying the boundary conditions*

$$\widehat{v}(m-2) = \widehat{v}(m-1)(p\lambda + q), \tag{3.14}$$

where $p = \delta\widetilde{c}(m-2)/\{(1-\delta K)[-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2)]\}$, $q = \widetilde{c}(m-2)[1 - \delta K - \delta\lambda_0]/\{(1-\delta K)[-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2)]\}$, *and* $K = z(m-1)/z(m-2)$. *Here,* $z(n)$ *is a solution to* (2.2) *for* $\lambda = \lambda_0$, *where* λ_0 *is less than the least eigenvalue of* (2.2), (3.2), *and* (3.13) *such that* $z(n) > 0$ *in the given interval,* $[-1, m-1]$.

Proof. Evaluating (3.1) at $n = m-1$ and $n = m-2$ gives the following:

$$\widehat{v}(m-1) = \widetilde{v}(m-1) - \widetilde{v}(m-2)\frac{z(m-1)}{z(m-2)}, \tag{3.15}$$

$$\widehat{v}(m-2) = \widetilde{v}(m-2) - \widetilde{v}(m-3)\frac{z(m-2)}{z(m-3)}. \tag{3.16}$$

By considering $\widetilde{v}(n)$ satisfying (2.2) at $n = m-2$, we obtain that

$$\widetilde{v}(m-3) = \left[\frac{\widetilde{b}(m-2)}{\widetilde{c}(m-3)} - \lambda\frac{\widetilde{c}(m-2)}{\widetilde{c}(m-3)}\right]\widetilde{v}(m-2) - \frac{\widetilde{c}(m-2)}{\widetilde{c}(m-3)}\widetilde{v}(m-1). \tag{3.17}$$

Substituting (3.17) into (3.16) gives the following:

$$\widehat{v}(m-2) = \widetilde{v}(m-2)\left\{1 - \left[\frac{\widetilde{b}(m-2)}{\widetilde{c}(m-3)} - \lambda\frac{\widetilde{c}(m-2)}{\widetilde{c}(m-3)}\right]\frac{z(m-2)}{z(m-3)}\right\} + \widetilde{v}(m-1)\frac{z(m-2)\widetilde{c}(m-2)}{z(m-3)\widetilde{c}(m-3)}. \tag{3.18}$$

Now using (3.13) together with (3.15) yields

$$\tilde{v}(m-1) = \frac{\hat{v}(m-1)}{1-\delta(z(m-1)/z(m-2))}, \tag{3.19}$$

which in turn, by substituting into (3.13), gives the following:

$$\tilde{v}(m-2) = \frac{\delta\hat{v}(m-1)}{1-\delta(z(m-1)/z(m-2))}. \tag{3.20}$$

Thus, by putting (3.19) and (3.20) into (3.18), we obtain

$$\begin{aligned}\hat{v}(m-2) = {}& \frac{\delta\hat{v}(m-1)}{1-\delta(z(m-1)/z(m-2))}\left\{1-\left[\frac{\tilde{b}(m-2)}{\tilde{c}(m-3)}-\lambda\frac{\tilde{c}(m-2)}{\tilde{c}(m-3)}\right]\frac{z(m-2)}{z(m-3)}\right\}\\ &+\frac{\hat{v}(m-1)z(m-2)\tilde{c}(m-2)}{1-\delta(z(m-1)/[z(m-2)z(m-3)\tilde{c}(m-3)])}.\end{aligned} \tag{3.21}$$

The equation above may be rewritten as follows:

$$\begin{aligned}&\left[1-\delta\frac{z(m-1)}{z(m-2)}\right]\hat{v}(m-2)\\ &\quad=\hat{v}(m-1)\left\{\frac{\delta\left[\tilde{c}(m-3)z(m-3)-\left(\tilde{b}(m-2)-\lambda\tilde{c}(m-2)\right)z(m-2)\right]+\tilde{c}(m-2)z(m-2)}{\tilde{c}(m-3)z(m-3)}\right\}.\end{aligned} \tag{3.22}$$

Now, since $z(n)$ is a solution to (2.2) for $\lambda = \lambda_0$, we have that

$$\tilde{c}(m-3)z(m-3) = -\tilde{c}(m-2)z(m-1)+\tilde{b}(m-2)z(m-2)-\lambda_0\tilde{c}(m-2)z(m-2). \tag{3.23}$$

Substituting (3.23) into (3.22) gives the following:

$$\begin{aligned}&\left[1-\delta\frac{z(m-1)}{z(m-2)}\right]\hat{v}(m-2)\\ &\quad=\hat{v}(m-1)\left\{\frac{-\delta\tilde{c}(m-2)z(m-1)+\delta(\lambda-\lambda_0)\tilde{c}(m-2)z(m-2)+\tilde{c}(m-2)z(m-2)}{-\tilde{c}(m-2)z(m-1)+\tilde{b}(m-2)z(m-2)-\lambda_0\tilde{c}(m-2)z(m-2)}\right\}.\end{aligned} \tag{3.24}$$

Setting $z(m-1)/z(m-2) = K$ yields

$$[1-\delta K]\hat{v}(m-2) = \hat{v}(m-1)\left\{\frac{-\delta\tilde{c}(m-2)K+\delta(\lambda-\lambda_0)\tilde{c}(m-2)+\tilde{c}(m-2)}{-\tilde{c}(m-2)K+\tilde{b}(m-2)-\lambda_0\tilde{c}(m-2)}\right\}. \tag{3.25}$$

Hence,

$$\widehat{v}(m-2)=\widehat{v}(m-1)\left\{\frac{\delta\widetilde{c}(m-2)\lambda+\widetilde{c}(m-2)(-\delta K-\delta\lambda_0+1)}{(1-\delta K)\left[-\widetilde{c}(m-2)K+\widetilde{b}(m-2)-\lambda_0\widetilde{c}(m-2)\right]}\right\}, \tag{3.26}$$

which is of the form (3.14), where $K=z(m-1)/z(m-2)$, $p=\delta\widetilde{c}(m-2)/\{(1-\delta K)[-K\widetilde{c}(m-2)+\widetilde{b}(m-2)-\lambda_0\widetilde{c}(m-2)]\}$, and $q=\widetilde{c}(m-2)[1-\delta K-\delta\lambda_0]/\{(1-\delta K)[-K\widetilde{c}(m-2)+\widetilde{b}(m-2)-\lambda_0\widetilde{c}(m-2)]\}$. □

Note that if we require that $a\lambda+b$ in (3.3) be a Nevanlinna or Herglotz function, then we must have that $a\geq 0$. This condition provides constraints on the allowable values of k.

Remark 3.3. In Theorems 3.1 and 3.2, we have taken $z(n)$ to be a solution of (2.2) for $\lambda=\lambda_0$ with λ_0 less than the least eigenvalue of (2.2), (3.2), and (3.13) such that $z(n)>0$ in $[-1,m-1]$. We assume that $z(n)$ does not obey the boundary conditions (3.2) and (3.13) which is sufficient for the results which we wish to obtain in this paper. However, this case will be dealt with in detail in a subsequent paper.

Theorem 3.4. *If $k=z(0)/z(-1)$ where $z(n)$ is a solution to (2.2) for $\lambda=\lambda_0$ with λ_0 less than the least eigenvalue of (2.2), (3.2), and (3.13) and $z(n)>0$ in the given interval $[-1,m-1]$, then the values of k which ensure that $a\geq 0$ in (3.3), that is, which ensure that $a\lambda+b$ in (3.3) is a Nevanlinna function are*

$$k\in\left(0,\frac{1}{\gamma}\right),\quad \text{for } \gamma>0. \tag{3.27}$$

Proof. From Theorem 3.1, we have that

$$a=\frac{\gamma k}{\widehat{c}(-1)/\widehat{c}(0)-k\gamma(\widehat{c}(-1)/\widehat{c}(0))}. \tag{3.28}$$

Assume that $\gamma>0$, then to ensure that $a\geq 0$ we require that either $k\geq 0$ and $\widehat{c}(-1)/\widehat{c}(0)-k\gamma(\widehat{c}(-1)/\widehat{c}(0))>0$ or $k\leq 0$ and $\widehat{c}(-1)/\widehat{c}(0)-k\gamma(\widehat{c}(-1)/\widehat{c}(0))<0$. For the first case, since $(\widehat{c}(-1)/\widehat{c}(0))\gamma>0$, we get $k\geq 0$ and $k<1/\gamma$. For the second case, we obtain $k\leq 0$ and $k>1/\gamma$, which is not possible. Thus, allowable values of k for $\gamma>0$ are

$$k\in\left(0,\frac{1}{\gamma}\right). \tag{3.29}$$

Since $k=z(0)/z(-1)\neq 0$. If $\gamma<0$, then we must have that either $k\leq 0$ and $\widehat{c}(-1)/\widehat{c}(0)-k\gamma(\widehat{c}(-1)/\widehat{c}(0))>0$ or $k\geq 0$ and $\widehat{c}(-1)/\widehat{c}(0)-k\gamma(\widehat{c}(-1)/\widehat{c}(0))<0$. The first case of $k\leq 0$ is not possible since $\widehat{c}(n)=(z(n-1)/z(n))\widetilde{c}(n-1)$ and $\widehat{c}(n)$, $\widetilde{c}(n-1)>0$, which implies that $z(n-1)/z(n)>0$ in particular for $n=0$. For the second case, we get $k\geq 0$ and $k<1/\gamma$ which is not possible. Thus for $\gamma<0$, there are no allowable values of k. □

Also, if we require that $p\lambda + q$ from (3.14) be a Nevanlinna/Herglotz function, then we must have $p \geq 0$. This provides conditions on the allowable values of K.

Corollary 3.5. *If $K = z(m-1)/z(m-2)$ where $z(n)$ is a solution to* (2.2) *for $\lambda = \lambda_0$ with λ_0 less than the least eigenvalue of* (2.2), (3.2), *and* (3.13), *and $z(n) > 0$ in the given interval $[-1, m-1]$, then*

$$K \in \left(-\infty, \frac{1}{\delta}\right) \cup \left(\frac{\widetilde{b}(m-2)}{\widetilde{c}(m-2)}, \infty\right), \quad \textit{for } \delta > \frac{\widetilde{c}(m-2)}{\widetilde{b}(m-2)},$$
$$K \in \left(-\infty, \frac{\widetilde{b}(m-2)}{\widetilde{c}(m-2)}\right) \cup \left(\frac{1}{\delta}, \infty\right), \quad \textit{for } \delta < \frac{\widetilde{c}(m-2)}{\widetilde{b}(m-2)}. \tag{3.30}$$

Proof. Without loss of generality, we may shift the spectrum of (2.2) with boundary conditions (3.2), (3.13), such that the least eigenvalue of (2.2) with boundary conditions (3.2), (3.13) is strictly greater than 0, and thus we may assume that $\lambda_0 = 0$.

Since $\widetilde{c}(m-2) > 0$, we consider the two cases, $\delta > 0$ and $\delta < 0$.

Assume that $\delta > 0$, then the numerator of p is strictly positive. Thus, to ensure that $p > 0$ the denominator must be strictly positive, that is, $(1-\delta K)[-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2)] > 0$. So either $1-\delta K > 0$ and $-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2) > 0$ or $1-\delta K < 0$ and $-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2) < 0$. Since $\lambda_0 = 0$, we have that either $K < 1/\delta$ and $K < \widetilde{b}(m-2)/\widetilde{c}(m-2)$ or $K > 1/\delta$ and $K > \widetilde{b}(m-2)/\widetilde{c}(m-2)$. Thus, if $1/\delta < \widetilde{b}(m-2)/\widetilde{c}(m-2)$, that is, $\delta > \widetilde{c}(m-2)/\widetilde{b}(m-2)$, we get

$$K \in \left(-\infty, \frac{1}{\delta}\right) \cup \left(\frac{\widetilde{b}(m-2)}{\widetilde{c}(m-2)}, \infty\right), \tag{3.31}$$

and if $1/\delta > \widetilde{b}(m-2)/\widetilde{c}(m-2)$, that is, $\delta < \widetilde{c}(m-2)/\widetilde{b}(m-2)$, we get

$$K \in \left(-\infty, \frac{\widetilde{b}(m-2)}{\widetilde{c}(m-2)}\right) \cup \left(\frac{1}{\delta}, \infty\right). \tag{3.32}$$

Now if $\delta < 0$, then the numerator of p is strictly negative. Thus, in order that $p > 0$, we require that the denominator is strictly negative, that is, $(1-\delta K)[-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2)] < 0$. So either $1-\delta K > 0$ and $-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2) < 0$ or $1-\delta K < 0$ and $-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2) > 0$. As $\lambda_0 = 0$, we obtain that either $K > 1/\delta$ and $K > \widetilde{b}(m-2)/\widetilde{c}(m-2)$ or $K < 1/\delta$ and $K < \widetilde{b}(m-2)/\widetilde{c}(m-2)$. These are the same conditions as we had on K for $\delta > 0$. Thus, the sign of δ does not play a role in finding the allowable values of K which ensure that $p \geq 0$, and hence we have the required result. □

4. Comparison of the Spectra

In this section, we see how the transformation, (3.1), affects the spectrum of the difference equation with various boundary conditions imposed at the initial and terminal points.

By combining the results of [1, conclusion] with Theorems 3.1 and 3.2, we have proved the following result.

Theorem 4.1. *Assume that* $\tilde{v}(n)$ *satisfies* (2.2)*. Consider the following four sets of boundary conditions:*

$$\tilde{v}(-1) = 0, \quad \tilde{v}(m-1) = 0, \tag{4.1}$$

$$\tilde{v}(-1) = 0, \quad \tilde{v}(m-2) = \delta\tilde{v}(m-1), \tag{4.2}$$

$$\tilde{v}(-1) = \gamma\tilde{v}(0), \quad \tilde{v}(m-1) = 0, \tag{4.3}$$

$$\tilde{v}(-1) = \gamma\tilde{v}(0), \quad \tilde{v}(m-2) = \delta\tilde{v}(m-1). \tag{4.4}$$

The transformation (3.1)*, where* $z(n)$ *is a solution to* (2.2) *for* $\lambda = \lambda_0$*, where* λ_0 *is less than the least eigenvalue of* (2.2) *with one of the four sets of boundary conditions above, such that* $z(n) > 0$ *in the given interval* $[-1, m-1]$*, takes* $\tilde{v}(n)$ *obeying* (2.2) *to* $\hat{v}(n)$ *obeying* (2.5)*.*

In addition,

(i) $\tilde{v}$ *obeying* (4.1) *transforms to* $\hat{v}$ *obeying*

$$\hat{h}\hat{v}(-1) + \hat{v}(0) = 0, \tag{4.5}$$

where $\hat{h} = [(\hat{c}(0)/\hat{c}(-1))(\tilde{b}(0)/\tilde{c}(0) - z(1)/z(0) - \hat{b}(0)/\hat{c}(0))]^{-1}$ *and*

$$\widehat{H}\hat{v}(m-1) + \hat{v}(m) = 0, \tag{4.6}$$

where $\widehat{H} = \tilde{b}(m-2)/\tilde{c}(m-2) - \hat{b}(m-1)/\hat{c}(m-1) - z(m-2)\hat{c}(m-2)/[z(m-1)\hat{c}(m-1)]$ *with* $\hat{c}(-1) = \tilde{c}(-1)$*.*

(ii) $\tilde{v}$ *obeying* (4.2) *transforms to* $\hat{v}$ *obeying* (4.5) *and* (3.14)*.*

(iii) $\tilde{v}$ *obeying*(4.3) *transforms to* $\hat{v}$ *obeying* (3.3) *and* (4.6)*.*

(iv) $\tilde{v}$ *obeying* (4.4) *transforms to* $\hat{v}$ *obeying* (3.3) *and* (3.14)*.*

The next theorem, shows that the boundary value problem given by $\tilde{v}(n)$ obeying (2.2) together with any one of the four types of boundary conditions in the above theorem has $m-1$ eigenvalues as a result of the eigencondition being the solution of an $(m-1)$th order polynomial in λ. It should be noted that if the boundary value problem considered is self-adjoint, then the eigenvalues are real, otherwise the complex eigenvalues will occur as conjugate pairs.

Theorem 4.2. *The boundary value problem given by* $\tilde{v}(n)$ *obeying* (2.2) *together with any one of the four types of boundary conditions given by* (4.1) *to* (4.4) *has* $m-1$ *eigenvalues.*

Proof. Since $\tilde{v}(n)$ obeys (2.2), we have that, for $n = 0, \ldots, m-2$,

$$\tilde{v}(n+1) = \frac{-\tilde{c}(n-1)\tilde{v}(n-1)}{\tilde{c}(n)} + \left(\frac{\tilde{b}(n)}{\tilde{c}(n)} - \lambda\right)\tilde{v}(n). \tag{4.7}$$

So setting $n = 0$, in (4.7), gives the following:

$$\tilde{v}(1) = \frac{-\tilde{c}(-1)\tilde{v}(-1)}{\tilde{c}(0)} + \left(\frac{\tilde{b}(0)}{\tilde{c}(0)} - \lambda\right)\tilde{v}(0). \tag{4.8}$$

For the boundary conditions (4.1) and (4.2), we have that $\tilde{v}(-1) = 0$ giving

$$\tilde{v}(1) = \left(\frac{\tilde{b}(0)}{\tilde{c}(0)} - \lambda\right)\tilde{v}(0) := \left(P_0^1 + P_1^1\lambda\right)\tilde{v}(0), \tag{4.9}$$

where P_0^1 and P_1^1 are real constants, that is, a first order polynomial in λ.

Also $n = 1$ in (4.7) gives that

$$\tilde{v}(2) = \frac{-\tilde{c}(0)\tilde{v}(0)}{\tilde{c}(1)} + \left(\frac{\tilde{b}(1)}{\tilde{c}(1)} - \lambda\right)\tilde{v}(1). \tag{4.10}$$

Substituting in for $\tilde{v}(1)$, from above, we obtain

$$\tilde{v}(2) = \left[\frac{-\tilde{c}(0)}{\tilde{c}(1)} + \left(\frac{\tilde{b}(1)}{\tilde{c}(1)} - \lambda\right)\left(\frac{\tilde{b}(0)}{\tilde{c}(0)} - \lambda\right)\right]\tilde{v}(0) := \left[P_0^2 + P_1^2\lambda + P_2^2\lambda^2\right]\tilde{v}(0), \tag{4.11}$$

where again $P_i^2,\ i = 0, 1, 2$ are real constants, that is, a quadratic polynomial in λ.

Thus, by an easy induction, we have that

$$\begin{aligned}\tilde{v}(m-1) &= \left[P_0^{m-1} + P_1^{m-1}\lambda + \cdots + P_{m-1}^{m-1}\lambda^{m-1}\right]\tilde{v}(0),\\ \tilde{v}(m-2) &= \left[P_0^{m-2} + P_1^{m-2}\lambda + \cdots + P_{m-2}^{m-2}\lambda^{m-2}\right]\tilde{v}(0),\end{aligned} \tag{4.12}$$

where $P_i^{m-1}, i = 0, 1, \ldots, m-1$ and $P_i^{m-2},\ i = 0, 1, \ldots, m-2$ are real constants, that is, an $(m-1)$th and an $(m-2)$th order polynomial in λ, respectively.

Now, (4.1) gives $\tilde{v}(m-1) = 0$, that is,

$$\left[P_0^{m-1} + P_1^{m-1}\lambda + \cdots + P_{m-1}^{m-1}\lambda^{m-1}\right]\tilde{v}(0) = 0. \tag{4.13}$$

So our eigencondition is given by

$$\left[P_0^{m-1} + P_1^{m-1}\lambda + \cdots + P_{m-1}^{m-1}\lambda^{m-1}\right] = 0, \tag{4.14}$$

which is an $(m-1)$th order polynomial in λ and, therefore, has $m-1$ roots. Hence, the boundary value problem given by $\tilde{v}(n)$ obeying (2.2) with (4.1) has $m-1$ eigenvalues.

Next, (4.2) gives $\tilde{v}(m-2) = \delta\tilde{v}(m-1)$, so

$$\left[P_0^{m-2} + P_1^{m-2}\lambda + \cdots + P_{m-2}^{m-2}\lambda^{m-2}\right]\tilde{v}(0) = \delta\left[P_0^{m-1} + P_1^{m-1}\lambda + \cdots + P_{m-1}^{m-1}\lambda^{m-1}\right]\tilde{v}(0), \tag{4.15}$$

from which we obtain the following eigencondition:

$$\left[P_0^{m-2} + P_1^{m-2}\lambda + \cdots + P_{m-2}^{m-2}\lambda^{m-2}\right] = \delta\left[P_0^{m-1} + P_1^{m-1}\lambda + \cdots + P_{m-1}^{m-1}\lambda^{m-1}\right]. \tag{4.16}$$

This is again an $(m-1)$th order polynomial in λ and therefore has $m-1$ roots. Hence, the boundary value problem given by $\tilde{v}(n)$ obeying (2.2) with (4.2) has $m-1$ eigenvalues.

Now for the boundary conditions (4.3) and (4.4), we have that $\tilde{v}(-1) = \gamma\tilde{v}(0)$, thus (4.8) becomes

$$\tilde{v}(1) = \left[\frac{-\tilde{c}(-1)}{\tilde{c}(0)} + \left(\frac{\tilde{b}(0)}{\tilde{c}(0)} - \lambda\right)\frac{1}{\gamma}\right]\tilde{v}(-1) := \left(Q_0^1 + Q_1^1\lambda\right)\tilde{v}(-1), \tag{4.17}$$

where Q_0^1 and Q_1^1 are real constants, that is, a first order polynomial in λ.

Using $\tilde{v}(-1) = \gamma\tilde{v}(0)$ and $\tilde{v}(1)$ from above, we can show that $\tilde{v}(2)$ can be written as the following:

$$\tilde{v}(2) := \left[Q_0^2 + Q_1^2\lambda + Q_2^2\lambda^2\right]\tilde{v}(-1), \tag{4.18}$$

where again Q_i^2, $i = 0, 1, 2$ are real constants, that is, a quadratic polynomial in λ.

Thus, by induction,

$$\begin{aligned}\tilde{v}(m-1) &= \left[Q_0^{m-1} + Q_1^{m-1}\lambda + \cdots + Q_{m-1}^{m-1}\lambda^{m-1}\right]\tilde{v}(-1),\\ \tilde{v}(m-2) &= \left[Q_0^{m-2} + Q_1^{m-2}\lambda + \cdots + Q_{m-2}^{m-2}\lambda^{m-2}\right]\tilde{v}(-1),\end{aligned} \tag{4.19}$$

where Q_i^{m-1}, $i = 0, 1, \ldots, m-1$ and Q_i^{m-2}, $i = 0, 1, \ldots, m-2$ are real constants, thereby giving an $(m-1)$th and an $(m-2)$th order polynomial in λ, respectively.

Now, (4.3) gives $\tilde{v}(m-1) = 0$, that is,

$$\left[Q_0^{m-1} + Q_1^{m-1}\lambda + \cdots + Q_{m-1}^{m-1}\lambda^{m-1}\right]\tilde{v}(-1) = 0. \tag{4.20}$$

So our eigencondition is given by

$$\left[Q_0^{m-1} + Q_1^{m-1}\lambda + \cdots + Q_{m-1}^{m-1}\lambda^{m-1}\right] = 0, \tag{4.21}$$

which is an $(m-1)$th order polynomial in λ and, therefore, has $m-1$ roots. Hence, the boundary value problem given by $\tilde{v}(n)$ obeying (2.2) with (4.3) has $m-1$ eigenvalues.

Lastly, (4.4) gives $\widetilde{v}(m-2) = \delta\widetilde{v}(m-1)$, that is,

$$\left[Q_0^{m-2} + Q_1^{m-2}\lambda + \cdots + Q_{m-2}^{m-2}\lambda^{m-2}\right]\widetilde{v}(-1) = \delta\left[Q_0^{m-1} + Q_1^{m-1}\lambda + \cdots + Q_{m-1}^{m-1}\lambda^{m-1}\right]\widetilde{v}(-1), \tag{4.22}$$

from which we obtain the following eigencondition:

$$\left[Q_0^{m-2} + Q_1^{m-2}\lambda + \cdots + Q_{m-2}^{m-2}\lambda^{m-2}\right] = \delta\left[Q_0^{m-1} + Q_1^{m-1}\lambda + \cdots + Q_{m-1}^{m-1}\lambda^{m-1}\right]. \tag{4.23}$$

This is again an $(m-1)$th order polynomial in λ and therefore has $m-1$ roots. Hence, the boundary value problem given by $\widetilde{v}(n)$ obeying (2.2) with (4.4) has $m-1$ eigenvalues. □

In a similar manner, we now prove that the transformed boundary value problems given in Theorem 4.1 have m eigenvalues, that is, the spectrum increases by one in each case.

Theorem 4.3. *The boundary value problem given by $\widehat{v}(n)$ obeying* (2.5)*, $n = 1, \ldots, m-2$, together with any one of the four types of transformed boundary conditions given in (i) to (iv) in Theorem 4.1 has m eigenvalues. The additional eigenvalue is precisely λ_0 with corresponding eigenfunction $z(n)$, as given in Theorem 4.1.*

Proof. The proof is along the same lines as that of Theorem 4.2. By Theorem 3.1, we have extended $\widehat{y}(n)$, such that $\widehat{y}(n)$ exists for $n = 1, \ldots, m-1$.

Since $\widehat{v}(n)$ obeys (2.5), we have that, for $n = 0, \ldots, m-2$,

$$\widehat{v}(n+1) = \frac{-\widehat{c}(n-1)\widehat{v}(n-1)}{\widehat{c}(n)} + \left(\frac{\widehat{b}(n)}{\widehat{c}(n)} - \lambda\right)\widehat{v}(n). \tag{4.24}$$

For the transformed boundary conditions in (i) and (ii) of Theorem 4.1, we have that (4.5) is obeyed, and as in Theorem 4.2, we can inductively show that

$$\begin{aligned}\widehat{v}(m-1) &= \left[M_0^{m-1} + M_1^{m-1}\lambda + \cdots + M_{m-1}^{m-1}\lambda^{m-1}\right]\widehat{v}(-1),\\ \widehat{v}(m-2) &= \left[M_0^{m-2} + M_1^{m-2}\lambda + \cdots + M_{m-2}^{m-2}\lambda^{m-2}\right]\widehat{v}(-1),\end{aligned} \tag{4.25}$$

and also by [1], we can extend the domain of $\widehat{v}(n)$ to include $n = m$ if necessary by forcing (4.6) and then

$$\widehat{v}(m) = \left[M_0^{m} + M_1^{m}\lambda + \cdots + M_{m}^{m}\lambda^{m}\right]\widehat{v}(-1), \tag{4.26}$$

where M_i^{m-1}, $i = 0, 1, \ldots, m-1$, M_i^{m-2}, $i = 0, 1, \ldots, m-2$, and M_i^m, $i = 0, 1, \ldots, m$ are real constants, that is, an $(m-1)$th, $(m-2)$th, and mth order polynomial in λ, respectively.

Now for (i), the boundary condition (4.6) gives the following:

$$\left[M_0^{m-1} + M_1^{m-1}\lambda + \cdots + M_{m-1}^{m-1}\lambda^{m-1}\right]\widehat{v}(-1) = \frac{-1}{\widehat{H}}\left[M_0^{m} + M_1^{m}\lambda + \cdots + M_{m}^{m}\lambda^{m}\right]\widehat{v}(-1). \tag{4.27}$$

Therefore, the eigencondition is

$$\left[M_0^{m-1} + M_1^{m-1}\lambda + \cdots + M_{m-1}^{m-1}\lambda^{m-1}\right] = \frac{-1}{\widehat{H}}\left[M_0^m + M_1^m\lambda + \cdots + M_m^m\lambda^m\right], \tag{4.28}$$

which is an mth order polynomial in λ and thus has m roots. Hence, the boundary value problem given by $\widehat{v}(n)$ obeying (2.5) with transformed boundary conditions (i), that is, (4.5) and (4.6), has m eigenvalues.

Also, for (ii), from the boundary condition (3.14), we get

$$\begin{aligned}&\left[M_0^{m-2} + M_1^{m-2}\lambda + \cdots + M_{m-2}^{m-2}\lambda^{m-2}\right]\widehat{v}(-1)\\ &\quad = (p\lambda + q)\left[M_0^{m-1} + M_1^{m-1}\lambda + \cdots + M_{m-1}^{m-1}\lambda^{m-1}\right]\widehat{v}(-1).\end{aligned} \tag{4.29}$$

Therefore, the eigencondition is

$$\left[M_0^{m-2} + M_1^{m-2}\lambda + \cdots + M_{m-2}^{m-2}\lambda^{m-2}\right] = (p\lambda + q)\left[M_0^{m-1} + M_1^{m-1}\lambda + \cdots + M_{m-1}^{m-1}\lambda^{m-1}\right], \tag{4.30}$$

which is an mth order polynomial in λ and thus has m roots. Hence, the boundary value problem given by $\widehat{v}(n)$ obeying (2.5) with transformed boundary conditions (ii), that is, (4.5) and (3.14), has m eigenvalues.

Putting $n = 0$ in (4.24), we get

$$\widehat{v}(1) = \frac{-\widehat{c}(-1)\widehat{v}(-1)}{\widehat{c}(0)} + \left(\frac{\widehat{b}(0)}{\widehat{c}(0)} - \lambda\right)\widehat{v}(0). \tag{4.31}$$

For the boundary conditions in (iii) and (iv), we have that (3.3) is obeyed, thus,

$$\widehat{v}(1) = \left(\frac{-\widehat{c}(-1)}{\widehat{c}(0)} + \left(\frac{\widehat{b}(0)}{\widehat{c}(0)} - \lambda\right)\frac{1}{a\lambda + b}\right)\widehat{v}(-1) := S_0^1 + \frac{1}{a\lambda + b}\left(R_0^1 + R_1^1\lambda\right), \tag{4.32}$$

where S_0^1, R_0^1, and R_1^1 are real constants.

Putting $n = 1$ in (4.24), we get

$$\widehat{v}(2) = \frac{-\widehat{c}(0)\widehat{v}(0)}{\widehat{c}(1)} + \left(\frac{\widehat{b}(1)}{\widehat{c}(1)} - \lambda\right)\widehat{v}(1), \tag{4.33}$$

which, by using (3.3) and $\widehat{v}(1)$, can be rewritten as follows:

$$\begin{aligned}\widehat{v}(2) &= \left[\frac{-\widehat{c}(-1)}{\widehat{c}(0)}\left(\frac{\widehat{b}(1)}{\widehat{c}(1)} - \lambda\right) + \left(\frac{-\widehat{c}(0)}{\widehat{c}(1)} + \left(\frac{\widehat{b}(1)}{\widehat{c}(1)} - \lambda\right)\left(\frac{\widehat{b}(0)}{\widehat{c}(0)} - \lambda\right)\right)\frac{1}{a\lambda + b}\right]\widehat{v}(-1)\\ &:= \left(\left(S_0^2 + S_1^2\lambda\right) + \left(R_0^2 + R_1^2\lambda + R_2^2\lambda^2\right)\frac{1}{a\lambda + b}\right)\widehat{v}(-1),\end{aligned} \tag{4.34}$$

where S_0^2, S_1^2, R_0^2, R_1^2, and R_2^2 are real constants.

Thus, inductively we obtain

$$\begin{aligned}
&\widehat{v}(m-1)\\
&\quad=\left(\left(S_0^{m-1}+S_1^{m-1}\lambda+\cdots+S_{m-2}^{m-1}\lambda^{m-2}\right)+\left(R_0^{m-1}+R_1^{m-1}\lambda+\cdots+R_{m-1}^{m-1}\lambda^{m-1}\right)\frac{1}{a\lambda+b}\right)\widehat{v}(-1),\\
&\widehat{v}(m-2)\\
&\quad=\left(\left(S_0^{m-2}+S_1^{m-2}\lambda+\cdots+S_{m-3}^{m-2}\lambda^{m-3}\right)+\left(R_0^{m-2}+R_1^{m-2}\lambda+\cdots+R_{m-2}^{m-2}\lambda^{m-2}\right)\frac{1}{a\lambda+b}\right)\widehat{v}(-1).
\end{aligned}\tag{4.35}$$

Also, by [1], we can again extend the domain of $\widehat{v}(n)$ to include $n=m$, if needed, by forcing (4.6), thus,

$$\widehat{v}(m)=\left(\left(S_0^{m}+S_1^{m}\lambda+\cdots+S_{m-1}^{m}\lambda^{m-1}\right)+\left(R_0^{m}+R_1^{m}\lambda+\cdots+R_{m}^{m}\lambda^{m}\right)\frac{1}{a\lambda+b}\right)\widehat{v}(-1),\tag{4.36}$$

where all the coefficients of λ are real constants.

The transformed boundary conditions (iii) mean that (4.6) is obeyed, thus, our eigencondition is

$$\begin{aligned}
&(a\lambda+b)\left(S_0^{m-1}+S_1^{m-1}\lambda+\cdots+S_{m-2}^{m-1}\lambda^{m-2}\right)+\left(R_0^{m-1}+R_1^{m-1}\lambda+\cdots+R_{m-1}^{m-1}\lambda^{m-1}\right)\\
&\qquad=\frac{-1}{\widehat{H}}\left[(a\lambda+b)\left(S_0^{m}+S_1^{m}\lambda+\cdots+S_{m-1}^{m}\lambda^{m-1}\right)+\left(R_0^{m}+R_1^{m}\lambda+\cdots+R_{m}^{m}\lambda^{m}\right)\right],
\end{aligned}\tag{4.37}$$

which is an mth order polynomial in λ and thus has m roots. Hence, the boundary value problem given by $\widehat{v}(n)$ obeying (2.5) with transformed boundary conditions (iii), that is, (3.3) and (4.6), has m eigenvalues.

Also, the transformed boundary conditions in (iv) give (3.14) which produces the following eigencondition:

$$\begin{aligned}
&(a\lambda+b)\left(S_0^{m-2}+S_1^{m-2}\lambda+\cdots+S_{m-3}^{m-2}\lambda^{m-3}\right)+\left(R_0^{m-2}+R_1^{m-2}\lambda+\cdots+R_{m-2}^{m-2}\lambda^{m-2}\right)\\
&\quad=(p\lambda+q)\left[(a\lambda+b)\left(S_0^{m-1}+S_1^{m-1}\lambda+\cdots+S_{m-2}^{m-1}\lambda^{m-2}\right)+\left(R_0^{m-1}+R_1^{m-1}\lambda+\cdots+R_{m-1}^{m-1}\lambda^{m}\right)\right],
\end{aligned}\tag{4.38}$$

which is an mth order polynomial in λ and thus has m roots. Hence, the boundary value problem given by $\widehat{v}(n)$ obeying (2.5) with transformed boundary conditions (iv), that is, (3.3) and (3.14), has m eigenvalues.

Lastly, we have that (3.1) transforms eigenfunctions of any of the boundary value problems in Theorem 4.2 to eigenfunctions of the corresponding transformed boundary value problem, see Theorem 4.2. In particular, if $\lambda_1<\cdots<\lambda_{m-1}$ are the eigenvalues of the original boundary value problem with corresponding eigenfunctions $\widetilde{u}_1(n),\ldots,\widetilde{u}_{m-1}(n)$, then $z(n),\widetilde{u}_1(n),\ldots,\widetilde{u}_{m-1}(n)$ are eigenfunctions of the corresponding transformed boundary value problem with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$. Since we know that the transformed boundary value problem has m eigenvalues, it follows that $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ constitute all the eigenvalues of the transformed boundary value problem, see [1]. □

5. Affine to Non-Dirichlet

In this section, we now show that the process in Section 3 may be reversed. In particular, by applying the following mapping:

$$v(n) = \hat{v}(n+1) - \hat{v}(n)\frac{u_0(n+1)}{u_0(n)}, \tag{5.1}$$

we can transform $\hat{v}$ obeying affine λ-dependent boundary conditions to v obeying non-Dirichlet boundary conditions.

Theorem 5.1. *Consider the boundary value problem given by $\hat{v}(n)$ satisfying* (2.5) *with the following boundary conditions:*

$$\hat{v}(-1) = \hat{v}(0)(\alpha\lambda + \beta), \tag{5.2}$$

$$\hat{v}(m-2) = \hat{v}(m-1)(\zeta\lambda + \eta). \tag{5.3}$$

The transformation (5.1)*, for $n = -1, \ldots, m-1$, where $u_0(n)$ is an eigenfunction of* (2.5), (5.2)*, and* (5.3) *corresponding to the eigenvalue $\lambda_0 = 0$, yields the following equation:*

$$c(n)v(n+1) - b(n)v(n) + c(n-1)v(n-1) = -c(n)\lambda v(n), \quad n = 0, \ldots, m-3, \tag{5.4}$$

where, for $\hat{c}(-1) = \tilde{c}(-1)$,

$$c(n) = \frac{u_0(n)\hat{c}(n)}{u_0(n+1)} > 0, \quad n = -1, \ldots, m-2,$$

$$b(n) = \left[\frac{u_0(n)\hat{c}(n)}{u_0(n+1)\hat{c}(n+1)} - \frac{\hat{c}(n-1)u_0(n-1)}{\hat{c}(n)u_0(n)} + \frac{\hat{b}(n)}{\hat{c}(n)} - \lambda_0\right]\frac{u_0(n)\hat{c}(n)}{u_0(n+1)}, \quad n = 0, \ldots, m-2. \tag{5.5}$$

In addition, $\hat{v}$ obeying (5.2) *and* (5.3) *transforms to v obeying the non-Dirichlet boundary conditions*

$$v(-1) = Bv(0), \tag{5.6}$$

$$v(m-2) = Av(m-1), \tag{5.7}$$

where $B = \alpha\hat{c}(0)/\{(\alpha\lambda_0 + \beta)[\hat{c}(0) + \alpha\hat{c}(-1)]\}$ and $A = [\eta(\hat{c}(m-2)/\hat{c}(m-1) + 1/\zeta)]^{-1}$.

Proof. The fact that $\hat{v}(n)$, obeying (2.5), transforms to $v(n)$, obeying (5.4), was covered in [1, conclusion]. Now, $\hat{v}$ is defined for $n = 0, \ldots, m-1$ and is extended to $n = -1, \ldots, m-1$ by

(5.2). Thus, v is defined for $n = -1, \dots, m-2$ giving that (5.4) is valid for $n = 0, \dots, m-3$. For $n = 0$ and $n = -1$, (5.1) gives the following:

$$v(0) = \hat{v}(1) - \hat{v}(0)\frac{u_0(1)}{u_0(0)}, \tag{5.8}$$

$$v(-1) = \hat{v}(0) - \hat{v}(-1)\frac{u_0(0)}{u_0(-1)}. \tag{5.9}$$

Setting $n = 0$ in (2.5) gives the following:

$$\hat{v}(1) = -\lambda\hat{v}(0) + \frac{\hat{b}(0)}{\hat{c}(0)}\hat{v}(0) - \frac{\hat{c}(-1)}{\hat{c}(0)}\hat{v}(-1), \tag{5.10}$$

which by using (5.2) becomes

$$\hat{v}(1) = \left[\frac{-\lambda\hat{c}(0) + \hat{b}(0) - \hat{c}(-1)(\alpha\lambda + \beta)}{\hat{c}(0)(\alpha\lambda + \beta)}\right]\hat{v}(-1). \tag{5.11}$$

Since $u_0(n)$ is an eigenfunction of (2.5), (5.2), and (5.3) corresponding to the eigenvalue $\lambda = \lambda_0 = 0$, we have that

$$\frac{u_0(-1)}{u_0(0)} = \alpha\lambda_0 + \beta, \tag{5.12}$$

and hence

$$u_0(1) = \left[\frac{-\lambda_0\hat{c}(0) + \hat{b}(0) - \hat{c}(-1)(\alpha\lambda_0 + \beta)}{\hat{c}(0)(\alpha\lambda_0 + \beta)}\right]u_0(-1). \tag{5.13}$$

Substituting (5.11) and (5.13) into (5.8) and using (5.2), we obtain

$$\begin{aligned} v(0) = {} & \left[\frac{-\lambda\hat{c}(0) + \hat{b}(0) - \hat{c}(-1)(\alpha\lambda + \beta)}{\hat{c}(0)(\alpha\lambda + \beta)}\right]\hat{v}(-1) \\ & - \frac{\hat{v}(-1)}{\alpha\lambda + \beta}\left[\frac{-\lambda_0\hat{c}(0) + \hat{b}(0) - \hat{c}(-1)(\alpha\lambda_0 + \beta)}{\hat{c}(0)(\alpha\lambda_0 + \beta)}\right]\frac{u_0(-1)}{u_0(0)}. \end{aligned} \tag{5.14}$$

Since $u_0(-1)/u_0(0) = \alpha\lambda_0 + \beta$, everything can be written over the common denominator $\hat{c}(0)(\alpha\lambda + \beta)$. Taking out $\hat{v}(-1)$ and simplifying, we get

$$v(0) = \hat{v}(-1)(\lambda_0 - \lambda)\frac{\hat{c}(0) + \alpha\hat{c}(-1)}{\hat{c}(0)(\alpha\lambda + \beta)}. \tag{5.15}$$

Thus,

$$\widehat{v}(-1)=v(0)\frac{\widehat{c}(0)(\alpha\lambda+\beta)}{(\lambda_0-\lambda)[\widehat{c}(0)+\alpha\widehat{c}(-1)]}. \tag{5.16}$$

Substituting (5.2) into (5.9) gives the following:

$$v(-1)=\widehat{v}(-1)\frac{\alpha(\lambda_0-\lambda)}{(\alpha\lambda+\beta)(\alpha\lambda_0+\beta)}. \tag{5.17}$$

Hence, by putting (5.16) into (5.17), we get

$$v(-1)=v(0)\left[\frac{\alpha\widehat{c}(0)}{(\alpha\lambda_0+\beta)[\widehat{c}(0)+\alpha\widehat{c}(-1)]}\right]. \tag{5.18}$$

So to impose the boundary condition (5.7), it is necessary to extend the domain of v by forcing the boundary condition (5.7). We must then check that v satisfies the equation on the extended domain.

Evaluating (5.4) at $n=m-2$ and using (5.7) give the following:

$$\left(\frac{1}{A}-\frac{b(m-2)}{c(m-2)}+\lambda\right)v(m-2)+\frac{c(m-3)}{c(m-2)}v(m-3)=0. \tag{5.19}$$

Using (5.1) with $n=m-2$ and $n=m-3$ together with (5.3), we obtain

$$\begin{aligned} v(m-2)&=\widehat{v}(m-1)-\widehat{v}(m-2)\frac{u_0(m-1)}{u_0(m-2)}=\widehat{v}(m-1)\left(1-(\zeta\lambda+\eta)\frac{u_0(m-1)}{u_0(m-2)}\right),\\ v(m-3)&=\widehat{v}(m-2)-\widehat{v}(m-3)\frac{u_0(m-2)}{u_0(m-3)}=\widehat{v}(m-1)(\zeta\lambda+\eta)-\widehat{v}(m-3)\frac{u_0(m-2)}{u_0(m-3)}. \end{aligned} \tag{5.20}$$

Substituting the above two equations into (5.19) yields

$$\begin{aligned} &\widehat{v}(m-1)\left[\left(\frac{1}{A}-\frac{b(m-2)}{c(m-2)}+\lambda\right)\left(1-(\zeta\lambda+\eta)\frac{u_0(m-1)}{u_0(m-2)}\right)+\frac{c(m-3)}{c(m-2)}(\zeta\lambda+\eta)\right]\\ &\quad-\widehat{v}(m-3)\frac{c(m-3)}{c(m-2)}\frac{u_0(m-2)}{u_0(m-3)}=0. \end{aligned} \tag{5.21}$$

Since $u_0(n)$ is an eigenfunction of (2.5), (5.2), and (5.3) corresponding to the eigenvalue $\lambda = \lambda_0 = 0$ we have that $u_0(m-2)/u_0(m-1) = \zeta\lambda_0 + \eta = \eta$. Thus, the above equation can be simplified to

$$-\widehat{v}(m-3) + \widehat{v}(m-1) \times \left[\zeta\lambda \frac{u_0(m-1)c(m-2)u_0(m-3)}{u_0(m-2)c(m-3)u_0(m-2)}\left(-\frac{1}{A} + \frac{b(m-2)}{c(m-2)} - \lambda\right) + (\zeta\lambda + \eta)\frac{u_0(m-3)}{u_0(m-2)}\right] = 0. \tag{5.22}$$

Also (2.5) evaluated at $n = m-2$ for $\widehat{y} = \widehat{v}$ together with (5.3) gives

$$\left(\frac{\widehat{c}(m-2)}{\widehat{c}(m-3)}\left(1 + \zeta\lambda^2 + \eta\lambda\right) - \frac{\widehat{b}(m-2)}{\widehat{c}(m-3)}(\zeta\lambda + \eta)\right)\widehat{v}(m-1) + \widehat{v}(m-3) = 0. \tag{5.23}$$

Adding (5.22) to (5.23) and using the fact that $\widehat{v}(m-1) \neq 0$ yields

$$\begin{aligned}&\zeta\lambda \frac{u_0(m-1)c(m-2)u_0(m-3)}{u_0(m-2)c(m-3)u_0(m-2)}\left(-\frac{1}{A} + \frac{b(m-2)}{c(m-2)} - \lambda\right) + (\zeta\lambda + \eta)\frac{u_0(m-3)}{u_0(m-2)}\\ &\quad + \frac{\widehat{c}(m-2)}{\widehat{c}(m-3)}\left(1 + \zeta\lambda^2 + \eta\lambda\right) - \frac{\widehat{b}(m-2)}{\widehat{c}(m-3)}(\zeta\lambda + \eta) = 0.\end{aligned} \tag{5.24}$$

By substituting in for $c(m-2)$ and $c(m-3)$, it is easy to see that all the λ^2 terms cancel out. Next, we examine the coefficients of λ^0, and using $u_0(m-2)/u_0(m-1) = \eta$, we obtain that the coefficient of λ^0 is

$$\frac{u_0(m-3)}{u_0(m-1)} + \frac{\widehat{c}(m-2)}{\widehat{c}(m-3)} - \frac{\widehat{b}(m-2)u_0(m-2)}{\widehat{c}(m-3)u_0(m-1)} \tag{5.25}$$

which equals 0 by (2.5) evaluated at $n = m-2$. Thus, only the terms in λ remain. First, we note that by substituting in for $c(m-2)$, $c(m-3)$ and $b(m-2)$ we get

$$\begin{aligned}\frac{u_0(m-1)c(m-2)u_0(m-3)}{u_0(m-2)c(m-3)u_0(m-2)} &= \frac{\widehat{c}(m-2)}{\widehat{c}(m-3)}\\ \frac{b(m-2)}{c(m-2)} &= \frac{u_0(m-2)\widehat{c}(m-2)}{u_0(m-1)\widehat{c}(m-1)} - \frac{\widehat{c}(m-3)u_0(m-3)}{\widehat{c}(m-2)u_0(m-2)} + \frac{\widehat{b}(m-2)}{\widehat{c}(m-2)}.\end{aligned} \tag{5.26}$$

Thus, equating coefficients of λ gives the following:

$$\frac{\widehat{c}(m-2)}{\widehat{c}(m-3)}\left(-\frac{\zeta}{A} + \zeta\frac{\widehat{c}(m-2)u_0(m-2)}{\widehat{c}(m-1)u_0(m-1)} + \eta\right) = 0. \tag{5.27}$$

Since $\widehat{c}(m-2)/\widehat{c}(m-3) \neq 0$, we can divide and solve for A to obtain

$$A = \left[\eta\left(\frac{\widehat{c}(m-2)}{\widehat{c}(m-1)} + \frac{1}{\zeta}\right)\right]^{-1}. \tag{5.28}$$

□

Note that the case of $\zeta = 0$, that is, a non-Dirichlet boundary condition, gives $A = 0$, that is, $v(m-2) = 0$ which corresponds to the results obtained in [1].

If we set $u_0(n) = 1/[z(n-1)\widetilde{c}(n-1)]$, with $z(n)$ a solution of (2.2) for $\lambda = \lambda_0 = 0$ where λ_0 less than the least eigenvalue of (2.2), (3.2), and (3.13) and $z(n) > 0$ in the given interval $[-1, m-1]$, then $u_0(n)$ is an eigenfunction of (2.5), (5.2), and (5.3) corresponding to the eigenvalue $\lambda_0 = 0$. To see that $u_0(n)$ satisfies (2.5), see [1, Lemma 4.1] with, as previously, $\widehat{l}u_0(1) = 0$, and now $u_0(-1) = \alpha\lambda_0 + \beta = \beta$. Then, by construction, $u_0(n)$ obeys (5.2). We now show that u_0 obeys (5.3). Let $K = z(m-1)/z(m-2)$,

$$\begin{aligned}
\zeta &= \frac{\delta\widetilde{c}(m-2)}{(1-\delta K)\left[-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2)\right]}, \\
\eta &= \frac{\widetilde{c}(m-2)[1-\delta K - \delta\lambda_0]}{(1-\delta K)\left[-K\widetilde{c}(m-2) + \widetilde{b}(m-2) - \lambda_0\widetilde{c}(m-2)\right]}.
\end{aligned} \tag{5.29}$$

Now $z(n)$ is a solution of (2.2) for $\lambda = \lambda_0$, thus,

$$\frac{u_0(m-2)}{u_0(m-1)} = \frac{z(m-2)\widetilde{c}(m-2)}{z(m-3)\widetilde{c}(m-3)} = \left[-\frac{z(m-1)}{z(m-2)} + \frac{\widetilde{b}(m-2)}{\widetilde{c}(m-2)} - \lambda_0\right]^{-1} = \zeta\lambda_0 + \eta = \eta. \tag{5.30}$$

Remark 5.2. For $u_0(n)$, α, β, ζ, and η as above, the transformation (5.1), in Theorem 5.1, results in the original given boundary value problem. In particular, we obtain that in Theorem 5.1 $c(n) = \widetilde{c}(n)$ and $b(n) = \widetilde{b}(n)$, see [1, Theorem 4.2]. In addition,

$$\begin{aligned}
B &= \frac{\alpha\widehat{c}(0)}{(\alpha\lambda_0 + \beta)[\widehat{c}(0) + \alpha\widehat{c}(-1)]} = \gamma, \\
A &= \left[\eta\left(\frac{\widehat{c}(m-2)}{\widehat{c}(m-1)} + \frac{1}{\zeta}\right)\right]^{-1} = \delta.
\end{aligned} \tag{5.31}$$

That is, the boundary value problem given by $\widehat{v}(n)$ satisfying (2.5) with boundary conditions (5.2), (5.3) transforms under (5.1) to $\widetilde{v}(n)$ obeying (2.2) with boundary conditions (3.2), (3.13) which is the original boundary value problem.

We now verify that $B = \gamma$. Let

$$\beta = \frac{\widehat{b}(0)/\widehat{c}(0) - \gamma(z(0)/z(-1))\left(\widehat{b}(0)/\widehat{c}(0)\right) - \widetilde{b}(0)/\widetilde{c}(0) + \gamma(\widetilde{c}(-1)/\widetilde{c}(0)) + z(1)/z(0)}{\widehat{c}(-1)/\widehat{c}(0) - \gamma(\widehat{c}(-1)/\widehat{c}(0))(z(0)/z(-1))},$$

$$\begin{aligned} \alpha &= \frac{\gamma(z(0)/z(-1))}{\widehat{c}(-1)/\widehat{c}(0) - \widehat{c}(-1)z(0)/[\widehat{c}(0)z(-1)]} \\ &= \frac{\gamma}{(\widehat{c}(-1)/\widehat{c}(0))(z(-1)/z(0)) - \widehat{c}(-1)/\widehat{c}(0)}. \end{aligned} \tag{5.32}$$

Since $\widehat{c}(-1) = \widetilde{c}(-1)$, we obtain $\widehat{c}(-1)/\widehat{c}(0) = z(0)/z(-1)$, and thus

$$\alpha = \frac{\gamma}{1 - \gamma(z(0)/z(-1))}. \tag{5.33}$$

Also, $B = \alpha\widehat{c}(0)/((\alpha\lambda_0 + \beta)\widehat{c}(0) + \alpha\widehat{c}(-1))$. Dividing through by $\alpha\widehat{c}(0)$ and using $\lambda_0 = 0$ together with $\widehat{c}(-1)/\widehat{c}(0) = z(0)/z(-1)$ gives the following:

$$\frac{1}{B} = \beta\left[\frac{1}{\alpha} + \frac{z(0)}{z(-1)}\right]. \tag{5.34}$$

Now,

$$\frac{\widehat{b}(0)}{\widehat{c}(0)} = \frac{z(-1)\widetilde{c}(-1)}{z(0)\widetilde{c}(0)} + \frac{z(0)}{z(-1)}, \tag{5.35}$$

and since z satisfies (2.2) at $n = 0$ for $\lambda = \lambda_0 = 0$, we get

$$\frac{\widetilde{b}(0)}{\widetilde{c}(0)} = \frac{z(-1)\widetilde{c}(-1)}{z(0)\widetilde{c}(0)} + \frac{z(1)}{z(0)}. \tag{5.36}$$

Thus, using (5.35) and (5.36), the numerator of β is simplified to

$$\frac{z(0)}{z(1)}\left(1 - \gamma\frac{z(0)}{z(1)}\right). \tag{5.37}$$

The denominator of β can be simplified using $\widehat{c}(-1)/\widehat{c}(0) = z(0)/z(-1)$ to

$$\frac{z(0)}{z(1)}\left(1 - \gamma\frac{z(0)}{z(1)}\right), \tag{5.38}$$

hence $\beta = 1$.

Finally, substituting in for α, we obtain

$$\left[\frac{1}{\alpha} + \frac{z(0)}{z(-1)}\right] = \frac{1}{\gamma}. \tag{5.39}$$

Thus, $1/B = 1/\gamma$, that is, $B = \gamma$.

Next, we show that $A = \delta$. Recall that $\lambda_0 = 0$ and

$$\frac{1}{A} = \eta\left(\frac{\widehat{c}(m-2)}{\widehat{c}(m-1)} + \frac{1}{\zeta}\right). \tag{5.40}$$

Let

$$\begin{gathered}\zeta = \frac{\delta\widetilde{c}(m-2)}{(1-\delta(z(m-1)/z(m-2)))\left[-(z(m-1)/z(m-2))\widetilde{c}(m-2) + \widetilde{b}(m-2)\right]},\\ \eta = \frac{\widetilde{c}(m-2)}{-(z(m-1)/z(m-2))\widetilde{c}(m-2) + \widetilde{b}(m-2)}.\end{gathered} \tag{5.41}$$

Note that

$$\frac{\widehat{c}(m-2)}{\widehat{c}(m-1)} = \frac{z(m-3)\widetilde{c}(m-3)z(m-1)}{z(m-2)z(m-2)\widetilde{c}(m-2)}, \tag{5.42}$$

and since z satisfies (2.2) at $n = m-2$ for $\lambda = \lambda_0 = 0$, we get

$$\frac{\widetilde{b}(m-2)}{\widetilde{c}(m-2)} = \frac{z(m-3)\widetilde{c}(m-3)}{z(m-2)\widetilde{c}(m-2)} + \frac{z(m-1)}{z(m-2)}. \tag{5.43}$$

We now substitute in for ζ and η into the equation for $1/A$ and use (5.42) and (5.43) to obtain that

$$\frac{1}{A} = \frac{1}{\delta}, \tag{5.44}$$

that is, $A = \delta$.

To summarise, we have the following.

Consider $\widehat{v}(n)$ obeying (2.5) with one of the following 4 types of boundary conditions:

(a) non-Dirichlet and non-Dirichlet, that is, (4.5) and (4.6);

(b) non-Dirichlet and affine, that is, (4.5) and (3.14);

(c) affine and non-Dirichlet, that is, (3.3) and (4.6);

(d) affine and affine, that is, (3.3) and (3.14).

By Theorem 4.3, each of the above boundary value problems have m eigenvalues.

Now, the transformation (5.1), with $u_0(n) = 1/z(n-1)\tilde{c}(n-1)$ an eigenfunction of (2.5) with boundary conditions (a) ((b), (c), (d), resp.) corresponding to the eigenvalue $\lambda = \lambda_0 = 0$, transforms $\widehat{v}(n)$ obeying (2.5) to $\tilde{v}(n)$ obeying (2.2) and transforms the boundary conditions as follows:

(1) boundary conditions (a) transform to $\tilde{v}(-1) = 0$ and $\tilde{v}(m-1) = 0$;

(2) boundary conditions (b) transform to $\tilde{v}(-1) = 0$ and (3.13);

(3) boundary conditions (c) transform to (3.2) and $\tilde{v}(m-1) = 0$;

(4) boundary conditions (d) transform to (3.2) and (3.13).

By Theorem 4.2, we know that the above transformed boundary value problems in $\tilde{v}(n)$ each have $m-1$ eigenvalues. In particular, if $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{m-1}$ are the eigenvalues of (2.5), (a) ((b), (c), (d), resp.) with eigenfunctions $u_0(n), \widehat{v}_1(n), \ldots, \widehat{v}_{m-1}(n)$, then $u_0(n) \equiv 0$ and $\widehat{v}_1(n), \ldots, \widehat{v}_{m-1}(n)$ are eigenfunctions of (2.2), (1) ((2), (3), (4), resp.) with eigenvalues $\lambda_1, \ldots, \lambda_{m-1}$. Since we know that these boundary value problems have $m-1$ eigenvalues, it follows that $\lambda_1, \ldots, \lambda_{m-1}$ constitute all the eigenvalues.

6. Conclusion

To conclude, we outline (the details are left to the reader to verify) how the entire process could also be carried out the other way around. That is, we start with a second order difference equation of the usual form, given in the previous sections, together with boundary conditions of one of the following forms:

(i) non-Dirichlet at the initial point and affine at the terminal point;

(ii) affine at the initial point and non-Dirichlet at the terminal point;

(iii) affine at the initial point and at the terminal point.

We can then transform the above boundary value problem (by extending the domain where necessary, as done previously) to an equation of the same type with, respectively, transformed boundary conditions as follows:

(A) Dirichlet at the initial point and non-Dirichlet at the terminal point;

(B) non-Dirichlet at the initial point and Dirichlet at the terminal point;

(C) non-Dirichlet at the initial point and at the terminal point.

It is then possible to return to the original boundary value problem by applying a suitable transformation to the transformed boundary value problem above.

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